CHAPTER I: FUNDAMENTALS

Section 1: Permutations

In this section you will learn about permutations. A ***permutation*** is simply a one-to-one, onto function from a nonempty set to itself. For example, the function $f:\{1,2,3,4\}\to\{1,2,3,4\}$ defined by $f(1)=2, f(2)=4, f(3)=3, \text{and} f(4)=1$ is a permutation on the set $\{1,2,3,4\}$. The set of all permutations on this set is usually denoted by S_4. Similarly, the set of all permutations on the set $\{1,2,3,\ldots,n\}$ is denoted by S_n.

The way I wrote the example permutation was a little bulky, and you can imagine that if the permutation was on the set of, say, 10 elements (or more), it could become quite laborious to write it in such a way. As we do often in mathematics, there is a shorthand version. I will illustrate it with an example.

Example 1: Suppose we have the above permutation $f:\{1,2,3,4\}\to\{1,2,3,4\}$ defined by $f(1)=2, f(2)=4, f(3)=3, \text{and} f(4)=1$. We can write this in the "two-line" format as follows: We form a 2×4 matrix in which the first row consists of the integers from 1 to 4. Beneath each element in the first row we write where that element is sent via the permutation. This forms the second row. So in the given example, the permutation on 4 elements would be written as $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$.

There is even a shorter version (called the ***cycle notation***) of this notation we will usually utilize. (Don't you love it? Mathematics has shorthand for the shorthand!) In this notation, you begin with the element 1 and follow it with where it is sent. In our example, that is 2. Then you next write where 2 is sent, namely 4. You continue until you get to the element that gets sent back to 1, which closes the notation. Notice that if an element is fixed by a given permutation, like 3 is in our example, it will not appear in cycle notation. If there are more elements remaining, we open another cycle and repeat until all non-fixed elements have been accounted for. For our example, we get the cycle $(1\,2\,4)$.

Example 2: Write the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 1 & 4 & 5 & 3 & 2 \end{pmatrix}$ in cycle notation.

Since 1 goes to 6, 6 goes to 3, and 3 goes back to 1, the first cycle we get is $(1\,6\,3)$. But there are elements unaccounted for. The first elements not yet dealt with is 2. In the given permutation, 2 gets sent to 7 and 7 goes back to 2. So now we have $(1\,6\,3)(2\,7)$. And that's it. Notice that since 4 and 5 are fixed by this permutation, they do not appear in the cycle notation.

Example 3: Write the two-line version of the permutation $(1\,7\,8\,2)(3\,5\,6)(4\,9)$.

Here it is: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 1 & 5 & 9 & 6 & 3 & 8 & 2 & 4 \end{pmatrix}$.

Note: In this last example, since the biggest number appearing in the cycle notation was 9, I assumed that this permutation was on the set $\{1,2,3,4,5,6,7,8,9\}$. But since fixed elements do not appear in the cycle notation, it is possible that the given permutation was on a larger set and those larger numbers were all fixed. Fortunately, that doesn't cause a problem in practice.

Example 4: Write the two-line version of the permutation $(1\,8\,2)(3\,7\,4)$.

Here it is: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 1 & 7 & 3 & 5 & 6 & 4 & 2 \end{pmatrix}$.

If every element is fixed under a given permutation (called the ***identity permutation***), we write its cycle notation as (1). We could just as well call it (2) or (3) or any singleton. But we use (1). If we have two permutations written back to back and wish to compute the "product" of the two, we perform the permutation on the right first. Here is an example.

Example 5: Compute $(1\,3\,4\,5)(1\,2\,4\,6)$.

If we "multiply" these together, beginning on the right, we see that 1 goes to 2, and then is fixed by the second permutation, so in the product 1 goes to 2. So we begin our answer by opening a cycle and writing $(1\,2$. Now, we determine where 2 goes. The element 2 goes to 4 in the right permutation which is then sent to 5 in the left permutation, so in the product 2 is sent to 5. Our product now looks like $(1\,2\,5$. The right permutation takes 5 to itself and then the left one sends 5 back to 1. This closes the cycle, giving us $(1\,2\,5)$. The first element not included is the element 3, so we open a new cycle with 3 and continue $(1\,2\,5)(3$. The right permutation sends 3 to itself and then the left one sends it to 4. The element 4 gets sent to 6, which then goes to itself. Finally, 6 is sent to 1, which is then sent to 3, closing the second cycle. The result is the product $(1\,2\,5)(3\,4\,6)$. This seems lengthy, but with practice it becomes quite simple.

Definition: Let p be a permutation. Cycles are disjoint if they have no numbers in common. If p is a single cycle of length k, we call it a k***-cycle***. The ***order*** of a permutation is the smallest power of the permutation that equals the identity.

Example 6: (a) Examples 2, 3, and 4 all involved disjoint cycles. The cycles in Example 5 are not disjoint since the number 4 appears in both cycles.

(b) $(1\,2\,5)$ is a 3-cycle.

(c) The order of $(1\,7\,8\,2)(3\,5\,6)(4\,9)$ is 12 since $\left[(1\,7\,8\,2)(3\,5\,6)(4\,9)\right]^{12} = (1)$. This is far from obvious, but we have the following facts.

Facts:(a) A k-cycle has order k.

(b) A product of disjoint cycles has order equal to the least common multiple of the lengths of the cycles.

So in our last example, I knew the order of $(1\,7\,8\,2)(3\,5\,6)(4\,9)$ was 12 since 12 is the least common multiple of 2, 3, and 4 (the lengths of the three cycles).

Theorem 1.1.1: Every permutation on the finite set $\{1,2,3,\ldots,n\}$ can be written uniquely as a product of disjoint cycles.

I will prove this theorem, but I will leave out some of the reasons. Your first exercise will be to supply the reasoning.

Proof: Let's do induction on the size of the set, n. If $n=1$, then there is only one permutation, the identity, and it is clearly the product of disjoint permutations (trivially). Now suppose that every permutation on *fewer* than n elements is a product of disjoint cycles. Let p be a non-identity permutation in S_n. Choose $x_0 \in \{1,2,3,\dots,n\}$ such that x_0 is not fixed by p. Define $x_1 = p(x_0)$, $x_2 = p(x_1)$, and so forth. There must be a number, k, such that $x_0, x_1, x_2, \dots, x_k$ are all distinct and $p(x_k) = x_0$. (Why?) Now consider these two subsets of $\{1,2,3,\dots,n\}$: $A = \{x_0, x_1, x_2, \dots, x_k\}$ and $B = \{1,2,3,\dots,n\} - A$. Note that $p(A) = A$ and $p(B) = B$. (Why?) Also note that if we restrict p to A, we get the k-cycle $(x_0\ x_1 \cdots x_k)$ and if we restrict p to B, we get a product of disjoint cycles. (Why?) Therefore, p is the product of disjoint cycles. (Why?)

Problem Set #1

1. Explain the proof of Theorem 1.1.1 by answering the four "Why?" questions.

2. List all the elements of S_3 (i.e. all permutations of three elements).

3. Convert each permutation from two-line format to cycle or vice versa.

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 2 & 6 & 7 & 5 & 4 \end{pmatrix}$ (b) $(1\,3\,4)(5\,6\,7)$

4. The inverse of a permutation p is the permutation, denoted by p^{-1}, such that $pp^{-1} = (1)$. Find the inverse of each permutation.

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 2 & 1 & 4 \end{pmatrix}$ (b) $(1\,4\,5)$

5. Notice that $(1\,4\,5) = (1\,5)(1\,4)$ and $(1\,3\,4\,6) = (1\,6)(1\,4)(1\,3)$. Show that any k-cycle can be written as a product of $(k-1)$ 2-cycles.